Solutions to tutorial exercises for stochastic processes

T1. (a) By the definition of the filtration we have $B_t \in \mathcal{F}_t$. Furthermore, since B_t is normally distributed, $\mathbb{E}|B_t| < \infty$, so that B_t is integrable for all $t \geq 0$. It remains to show that $\mathbb{E}[B_t - B_s|\mathcal{F}_s] = 0$. Because B has independent increments we know that $B_t - B_s$ is independent of $\mathcal{F}_s^0$ for all $0 \leq s \leq t$. Using Proposition 2.7 we find

$$\mathbb{E}[B_t - B_s|\mathcal{F}_s] = \mathbb{E}\left[B_t - B_s \,\middle|\, \mathcal{F}_s^0\right] = 0.$$

(b) By definition $N_t \in \mathcal{F}_t$ and

$$\mathbb{E}|N_t| \leq \mathbb{E}[B_t^2] + t = 2t < \infty.$$

Again using Proposition 2.7 we find for any $0 \leq s \leq t$

$$\begin{aligned}\mathbb{E}[N_t - N_s \mid \mathcal{F}_s] &= \mathbb{E}\left[B_t^2 - B_s^2 - t + s \,\middle|\, \mathcal{F}_s^0\right] \\ &= \mathbb{E}\left[(B_t - B_s)^2 + 2B_sB_t - 2B_s^2 \,\middle|\, \mathcal{F}_s^0\right] - t + s.\end{aligned}$$

Since $B_s \in \mathcal{F}_s^0$ and $B_t - B_s$ is independent of $\mathcal{F}_s^0$, we find

$$\begin{aligned}\mathbb{E}[N_t - N_s \mid \mathcal{F}_s] &= \mathbb{E}\big[(B_t - B_s)^2\big] + 2B_s\mathbb{E}[B_t \mid \mathcal{F}_s] - 2B_s^2 - t + s \\ &= t - s + 2B_s^2 - 2B_s^2 - t + s = 0.\end{aligned}$$

T2. Let $0 \leq t_1 < \cdots < t_n$. The vector $(X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$ has a multivariate normal distribution. Therefore it suffices to show that the increments are mutually uncorrelated. Note that since X is martingale, all increments have expectation 0. Let $1 < i < j \leq n$, then by using the law of total expectation we find

$$\begin{aligned}\mathrm{Cov}(X_{t_i} - X_{t_i-1}, X_{t_j} - X_{t_j-1}) &= \mathbb{E}[(X_{t_i} - X_{t_{i-1}})(X_{t_j} - X_{t_{j-1}})] \\ &= \mathbb{E}\big[\mathbb{E}[(X_{t_i} - X_{t_i-1})(X_{t_j} - X_{t_j-1}) \mid \mathcal{F}_{t_{j-1}}]\big] \\ &= \mathbb{E}\big[(X_{t_i} - X_{t_i-1})\mathbb{E}[X_{t_j} - X_{t_j-1} \mid \mathcal{F}_{t_{j-1}}]\big] \\ &= 0.\end{aligned}$$

T3. We use the law of total expectation to get

$$\begin{aligned}\mathbb{E}\big[(M_t - M_s)^2 \bigm| \mathcal{F}_r\big] &= \mathbb{E}\big[M_t^2 + M_s^2 \bigm| \mathcal{F}_r\big] - \mathbb{E}\big[2M_tM_s \bigm| \mathcal{F}_r\big] \\ &= \mathbb{E}\big[M_t^2 + M_s^2 \bigm| \mathcal{F}_r\big] - 2\mathbb{E}\Big[\mathbb{E}\big[M_tM_s \bigm| \mathcal{F}_s\big] \Bigm| \mathcal{F}_r\Big] \\ &= \mathbb{E}\big[M_t^2 + M_s^2 \bigm| \mathcal{F}_r\big] - 2\mathbb{E}\Big[M_s\mathbb{E}\big[M_t \bigm| \mathcal{F}_s\big] \Bigm| \mathcal{F}_r\Big] \\ &= \mathbb{E}\big[M_t^2 - M_s^2 \bigm| \mathcal{F}_r\big].\end{aligned}$$

T4. To use the stopping theorem we first need to show that $\mathbb{E}[\tau] < \infty$. We can define another stopping time by taking $\tau \wedge t$, so that $\mathbb{E}[\tau \wedge t] < \infty$ for all $t > 0$. We can use the stopping theorem on this stopping time to find

$$\mathbb{E}[\tau \wedge t] = \mathbb{E}[B_{\tau \wedge t}^2] \leq a^2 \vee b^2,$$

for all $t > 0$. Using Fatou's lemma we get

$$\mathbb{E}[\tau] \leq \liminf_{t \to \infty} \mathbb{E}[\tau \wedge t] \leq a^2 \vee b^2 < \infty.$$

We can now use the stopping theorem on τ to get

$$0 = B_0 = \mathbb{E}[B_\tau] = -a\mathbb{P}(B_\tau = -a) + b\mathbb{P}(B_\tau = b).$$

Combining this with the fact that $\mathbb{P}(B_\tau = -a) + \mathbb{P}(B_\tau = b) = 1$, we conclude that $\mathbb{P}(B_\tau = -a) = \frac{b}{a+b}$ and $\mathbb{P}(B_\tau = b) = \frac{a}{a+b}$. Furthermore, again using the stopping theorem:

$$\mathbb{E}[\tau] = \mathbb{E}[B_\tau^2] = a^2\mathbb{P}(B_\tau = -a) + b^2\mathbb{P}(B_\tau = b) = ab.$$

T5. Let $p := \mathbb{P}(\{v\})$, and thus $\mathbb{P}(\{w\}) = 1 - p$. Since the distribution has mean u, we find

$$vp + w(1 - p) = u.$$

It follows that

$$p = \frac{u - w}{v - w},$$

so that $\mathbb{P}$ is uniquely determined.